Digital Signal Processing
Lecture 2

1 Summary

This lecture will focus on:

1. Linear time invariant systems
2. Linear constant coefficient difference equations
3. Frequency domain representation of discrete time signals and systems

2 Linear time invariant systems

Linear time invariant systems represent a special class of discrete time systems. We have seen that any discrete signal can be expressed in terms of the unit sample. We have

$$x[n] = \sum_{k=-\infty}^{k=\infty} x[k]\delta[n-k]$$

The output of a linear system in response to $x[n]$ can be written as

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]T\{\delta[n-k]\}$$

where $T\{\delta[n-k]\}$ is the response of the linear system to $\delta[n-k]$. If $h[n]$ is the response of the linear system to $\delta[n]$ and the system is also time invariant we have

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$$

which is known as the convolution sum and is analogous to the convolution integral for continuous time systems. We sometimes use the notation $y[n] = x[n] * h[n]$.

The sequence $h[n]$ is known as the impulse response of the linear time invariant system. An LTI system is completely characterized by its impulse response.

2.1 Examples

1. A discrete time LTI system has an impulse response given by

$$h[n] = \begin{cases} 2 & n = -1 \\ 1 & n = 0 \\ 4 & n = 1 \\ -2 & n = 2 \\ 2 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Sketch $h[n]$ in the interval $-5 \leq n \leq 5$.

   (b) Compute the output of the system $y[n]$ when the input to the system is $x[n] = 2\delta[n] - \delta[n-1] + 4\delta[n-3]$.

2. Consider an LTI system with impulse response $h[n] = u[n] - u[n-N]$ where $N > 0$. Find the output of the system in response to the input $x[n] = u[n] - u[n-N]$. $u[n]$ is the unit step sequence and $N > 0$

2.2 Properties of LTI systems

The properties of LTI systems are derived from the properties of discrete convolution.

- The roles of the input and impulse response can be reversed. This follows from the fact that convolution is commutative
$$x[n] * h[n] = h[n] * x[n]$$
- Cascade systems. When the output of a system is the input to another system connected in cascade, the overall impulse response is the convolution of the two impulse responses.
- Parallel systems. Here the systems have the same input and the overall output is the sum of the two outputs. The overall impulse response is the sum of the two impulse responses.
- An LTI system is BIBO stable if and only if the impulse response is absolutely summable.
$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$
- For a causal system, $h[n] = 0$ for $n < 0$.

Exercises

Determine the impulse response for the following systems

1. An ideal delay where $y[n] = x[n - n_d]$ where n_d is a positive integer.
2. The moving average system where
$$y[n] = \frac{1}{M_1 + M2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

3. The accumulator system where

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

The impulse responses of the ideal delay system and the moving average system have only a finite number of non zero samples, such systems are called finite impulse response (FIR) systems.

On the other hand an impulse response with an infinite number of non zero samples is an infinite impulse response.

3 Linear constant coefficient difference equations

There are LTI systems where the input output relationship satisfies an Nth order linear constant coefficient difference equation. That is

$$\sum_{k=-0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

For example for an accumulator where

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

we can show that $y[n] = x[n] + y[n-1]$.

4 Frequency domain representation of discrete time signals and systems

A large class of signals can be represented as a linear combination of complex exponentials

$$x[n] = \sum_{k} \alpha_k e^{j\omega_k n}$$

Let us consider the response of an LTI system with impulse response $h[n]$ to the sequence $x[n] = e^{j\omega n}$ (a complex exponential with radian frequency ω). We have

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \end{aligned}$$

We define $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$ which yields

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

Thus $e^{j\omega n}$ is an eigenfunction of the system with the associated eigenvalue $H(e^{j\omega})$ which is known as the frequency response of the system.

If the input can be expressed as a linear combination of complex exponentials, that is

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}$$

Then the output is given by

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k}) e^{j\omega_k n}$$

Unlike the frequency response of a continuous time system, $H(e^{j\omega})$ is periodic with period 2π. Therefore we only need to represent $H(e^{j\omega})$ over an interval of 2π. We usually use the interval $-\pi \leq \omega \leq \pi$. Low frequencies correspond to those around 0 while high frequencies correspond to those around $\pm\pi$

4.1 Filters

- Low pass filters
- High pass filters
- Bandpass filters

4.2 Examples

We can show that

- For the ideal delay system $H(e^{j\omega}) = e^{j\omega n_d}$
- For the moving average system

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2}$$