Digital Signal Processing

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1 Summary

This lecture will focus on:

1. The z-transform

2 z-Transform

The z-transform of a discrete time sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Here, z is a complex variable and the z-transform is an infinite sum. Recall that the Fourier transform of a discrete time sequence $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

We see that if $z = e^{j\omega}$ the z-transform reduces to the Fourier transform. This corresponds to setting $|z| = 1$. That is the Fourier transform is the z-transform evaluated on the unit circle.

2.1 The Region of Convergence

The z-transform does not converge for all sequences or for all values of z. The values of z for which the z-transform converges for a particular sequence is known as the region of convergence (ROC). The ROC is a ring in the z-plane. If the ROC includes the unit circle, then the Fourier transform converges. It is useful when a closed form expression for the z-transform can be found and $X(z)$ can be expressed as a rational function

$$X(z) = \frac{P(z)}{Q(z)}$$

with $P(z)$ and $Q(z)$ both polynomials in z. The values of z for which $X(z) = 0$ are known as the zeros of $X(z)$ while the values of z for which $X(z)$ is infinite are known as the poles of $X(z)$.

2.1.1 Examples

Compute the z transform of the following sequences clearly indicating the region of convergence

1. $\delta[n]$
2. $u[n]$
3. $-u[-n-1]$
4. $x[n] = a^n u[n]$
5. $x[n] = -a^n u[-n-1]$

2.1.2 Properties of the Region of Convergence

1. The ROC is a ring in the z-plane centered at the origin: $0 \leq r_R < |z| < r_L \leq \infty$
2. The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle
3. The ROC cannot contain any poles
4. If $x[n]$ is a finite duration sequence then the ROC is the entire z-plane with the possible exception of $z = 0$ or $z = \infty$
5. If $x[n]$ is a right sided sequence then the ROC extends outward from the outermost finite pole in $X(z)$.
6. If $x[n]$ is a left sided sequence then the ROC extends inward from the innermost nonzero pole in $X(z)$.
7. If $x[n]$ is a two sided sequence then the ROC will consist of a ring in the z-plane bounded on the interior and exterior by a pole.

2.2 The Inverse z-transform

2.2.1 By Inspection

If the z-transform is already known and tabulated, then we can determine the sequence by inspection.

2.2.2 Partial Fraction Expansion

Let $X(z)$ be expressed as a ratio of polynomials in z^{-1}

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

We can also write

$$X(z) = \frac{a_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{b_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where the c_k's are the nonzero zeros of $X(z)$ and the d_k's are the nonzero poles of $X(z)$. If $M < N$ and all the poles are of first order, then we can write

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

This is the partial fraction expansion of $X(z)$ and the coefficients can be found as

$$A_k = (1 - d_k z^{-1}) X(z)|_{z=d_k}$$

The corresponding sequence can now be found by table lookup.

Example: Consider the sequence whose z-transform is

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$$