Signals and Communication Lecture 2

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1 Summary

This lecture will focus on:

1. Linear systems
2. Linear time invariant systems
3. Convolution

2 Linear systems

Linear systems satisfy two properties namely

1. Superposition: if input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces output $y_2(t)$. Then the output of the system in response to input $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. Homogeneity: If input $x(t)$ produces output $y(t)$, then input $ax(t)$ where $a \in \mathbf{C}$ produces output $ay(t)$.

A system is said to be time invariant if a delay in the input produces the same delay in the output. Also, the properties of the system do not change with time and therefore the response does not depend on when the input is applied. Formally, if the response to $x(t)$ is $y(t)$, then the response to $x(t-\tau)$ is $y(t-\tau)$ where τ is any real number.

A system that is both linear and time invariant is known as a linear time invariant system. Several systems including communication systems can be modelled as linear time invariant (LTI) systems. An LTI system is completly characterised by its impulse response $h(t)$. The impulse response of an LTI system is the response of the system to the Dirac delta function $\delta(t)$.

2.1 The Convolution Integral

In continuous time, the output $y(t)$ of an LTI system in response to input $x(t)$ is given by the convolution integral. That is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1}$$

Consider the pulse

$$p_{\Delta\tau}(t) = \begin{cases} \frac{1}{\Delta\tau} & |t| < \frac{\Delta\tau}{2} \\ 0 & \text{Otherwise} \end{cases}$$

We can show that for any continuous time signal $x(t)$,

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau) p_{\Delta\tau}(t - n\Delta\tau)\Delta\tau$$

Since as $\Delta\tau \to 0$, $p_{\Delta\tau}(t) \to \delta(t)$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This also follows by applying the sifting property of the Dirac delta function and exploiting the fact that the Dirac delta function is even.

If $x(t)$ is the input to an LTI system, we have

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau) R\{p_{\Delta\tau}(t - n\Delta\tau)\}\Delta\tau$$

where $R\{p_{\Delta\tau}(t - n\Delta\tau)\}$ is the response of the LTI system to the pulse $p_{\Delta\tau}(t - n\Delta\tau)$.

As $\Delta\tau \to 0$, $R\{p_{\Delta\tau}(t)$ tends to the response of the system to the Dirac delta function which we denote as $h(t)$. This is known as the impulse response of the system. Thus taking the limit as $\Delta\tau \to 0$ we get the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

We can show that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

2.1.1 Examples

1. A linear time invariant (LTI) system has the impulse response $h(t)$ shown in Figure 3. Determine and sketch the response of the system to the following inputs

   (a) $x(t) = \delta(t+2) + \delta(t-2)$

   (b) $x(t) = \delta(t+1) + \delta(t-1)$

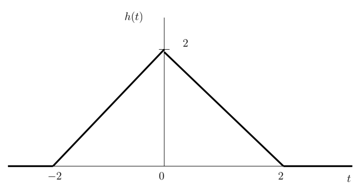

Figure 1:

2. Consider a linear time invariant (LTI) system with impulse response $h(t) = e^{-at}u(t)$ where $a > 0$ and $u(t)$ is the unit step function. Compute and sketch the response $y(t)$ of the LTI system to the following inputs $x(t)$

   (a) $x(t) = \delta(t)$ where $\delta(t)$ is the Dirac delta function.

   (b) $x(t) = u(t)$ where $u(t)$ is the unit step function.

Homework 2

1. Using relevant equations, explain what you understand by the term superposition when applied to linear systems.

2. Sketch the function $r(t) - r(t-T) - r(t-2T) + r(t-3T)$ where $r(t)$ is the unit ramp and $T > 0$.

3. If the pulse $p(t)$ below is passed through a linear time invariant system whose impulse response is $h(t) = u(t) - u(t-T)$ where $u(t)$ is the unit step function compute and sketch the output $y(t)$.