Signals and Communication Lecture 4

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1 Summary

This lecture will focus on:

1. Introduction to the Fourier transform
2. Properties of the Fourier transform
3. Fourier transform of periodic signals
4. Frequency response
5. Filters and bandwidth

2 The Fourier Transform

We have seen that the Fourier series allows us to represent a periodic signal as a sum of complex exponentials. In this section we introduce the Fourier transform which allows the representation of nonperiodic signals as a continuous sum of complex exponentials with frequencies ranging from $-\infty$ to $+\infty$. If a signal $x(t)$ satisfies the Dirichlet conditions

1. $x(t)$ is absolutely integrable that is $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
2. The number of minima and maxima of $x(t)$ in any finite interval on the real line is finite
3. The number of discontinuities of $x(t)$ in any finite interval on the real line is finite

Then we define the Fourier transform of $x(t)$ as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

We can obtain $x(t)$ from $X(f)$ via the inverse Fourier transform.

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$x(t)$ and $X(f)$ form a Fourier transform pair and we use the following notation $X(f) = \mathcal{F}[x(t)]$, similarly the inverse Fourier transform is denoted by $x(t) = \mathcal{F}^{-1}[X(f)]$

Example: Consider the rectangular pulse defined as follows

$$p(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

We showed in class that $\mathcal{F}[p(t)] = \frac{\sin(\pi f)}{(\pi f)} \triangleq \text{sinc}(f)$.
Example: Compute $\mathcal{F}[\delta(t)]$.

3 Properties of the Fourier Transform

1. Linearity: If $\mathcal{F}[x_1(t)] = X_1(f)$ and $\mathcal{F}[x_2(t)] = X_2(f)$ then $\mathcal{F}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 X_1(f) + \alpha_2 X_2(f)$. Where $\alpha_{1,2}$ are arbitrary scalars.

2. Time shifting: $\mathcal{F}[x(t - t_0)] = X(f)e^{-j2\pi f t_0}$

3. Time scaling: For any $a \neq 0$
$$\mathcal{F}[x(at)] = \frac{1}{|a|} X(\frac{f}{a})$$

4. Duality: If $\mathcal{F}[x(t)] = X(f)$ then $\mathcal{F}[X(t)] = x(-f)$

   Example: Show $\mathcal{F}[1] = \delta(f)$.

5. Frequency shifting (modulation): $\mathcal{F}[x(t)e^{j2\pi f_c t}] = X(f - f_c)$

   Example: Show $\mathcal{F}[e^{j2\pi f_c t}] = \delta(f - f_c)$

6. Convolution in time.
$$\mathcal{F}[\int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau] = H(f)X(f)$$

7. Differentiation in the time domain
$$\mathcal{F}[\frac{d}{dt} x(t)] = j2\pi f X(f)$$

8. Integration in the time domain
$$\mathcal{F}[\int_{-\infty}^{t} x(\tau)d\tau] = \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f)$$

9. Conjugate functions. If $\mathcal{F}[x(t)] = X(f)$ then $\mathcal{F}[x^*(t)] = X^*(-f)$

10. Multiplication in the time domain. If $\mathcal{F}[x_1(t)] = X_1(f)$ and $\mathcal{F}[x_2(t)] = X_2(f)$ then
$$\mathcal{F}[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} X_1(\lambda)X_2(f - \lambda)d\lambda$$

11. Rayleigh's energy theorem
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

4 Signal transmission over LTI systems

We have seen that the output $y(t)$ of an LTI system in response to $x(t)$ is given by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Also, the response of the LTI system to a complex exponential $e^{j2\pi ft}$ is a scaled version of the complex exponential $H(f)e^{j2\pi ft}$ where

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$$

.

The quantity $H(f)$ is known as the *frequency response* of the LTI system and it is the Fourier transform of the impulse response. In general, $H(f)$ is a complex number and we can write

$$H(f) = |H(f)|e^{j\theta(f)}$$

where $|H(f)|$ is known as the magnitude spectrum and $\theta(f)$ is known as the phase spectrum.

Using the convolution property of the Fourier transform we have $Y(f) = H(f)X(f)$. And $H(f)$ is also known as the *transfer function* of the system as it relates the Fourier transform of the output to that of the input.

4.1 Spectra of a real valued signal

For a real valued signal $x(t)$ with Fourier transform $X(f) = |X(f)|e^{j\theta(f)}$, the following holds

$$X(-f) = X^*(f)$$

Therefore

$$|X(-f)| = |X(f)|$$

and

$$\theta(-f) = -\theta(f)$$

That is the magnitude spectrum is an even function of frequency while the phase spectrum is an odd function of frequency.

4.2 System properties

Since the impulse response $h(t)$ completely characterizes an LTI system, system properties can be determined from $h(t)$. A system is *causal* if it doesn't respond before an excitation is applied. For a causal system,

$$h(t) = 0, \quad t < 0$$

Also, we showed in class that for a BIBO stable LTI system,

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty.$$

5 Filters and Bandwidth

Filters are frequency selective systems that limit the frequency content of input signals to within a specified range. Frequency components of the input signal that are present in the output lie within the passband of the filter. Frequency components that are present in the input but absent from the output lie in the stopband of the filter.

The bandwidth of a filter is a number used to measure the extent of significant frequency content (in the positive frequency range) that is allowed to pass through the filter. Consider an ideal lowpass filter with a frequency response given by

$$H_{LP}(f) = \begin{cases} 1 & |f| \leq W \\ 0 & \text{otherwise} \end{cases}$$

The bandwidth of this filter is W.

Consider the ideal bandpass filter given by

$$H_{BP}(f) = \begin{cases} 1 & W_1 \leq |f| \leq W_2 \\ 0 & \text{otherwise} \end{cases}$$

The bandwidth of this filter is $W_2 - W_1$.

The frequency response of an ideal highpass filter is given by

$$H_{HP}(f) = \begin{cases} 1 & |f| \geq W \\ 0 & \text{otherwise} \end{cases}$$

5.1 3dB Bandwidth

For a non-ideal low pass filter with maximum magnitude response at $f = 0$, The 3dB bandwidth is the frequency at which the magnitude response is $H(0)/\sqrt{2}$. At this frequency the power transfer is half that at the origin. The power transfer is proportional to the square of the magnitude response.

Example: Consider a system with impulse reponse $h(t) = e^{-t}u(t)$

Homework 3

1. Read chapter 3 of John G. Proakis and Masoud Salehi *Fundamentals of Communication Systems*, Pearson Education on Amplitude modulation.

2. Prove Parsavel's relation relating the power content of a periodic signal to the sum of the power content of its frequency components. That is

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

3. Prove Rayleigh's energy theorem for Fourier transforms