Signals and Communication Lecture 6

Ciira wa Maina, Ph.D.
ciira.maina@dkut.ac.ke

1 Summary

This lecture will focus on:

1. Introduction to Angle Modulation
2. Frequency modulation
3. Narrowband and Wideband FM
4. Transmission bandwidth of FM signals
5. Generation of FM Signals

2 Introduction to Angle Modulation

As discussed when we introduced amplitude modulation, modulation involves modifying some property of the carrier signal by the baseband message signal $m(t)$. In general the carrier is given by

$$c(t) = A_c \cos(\underbrace{2\pi f_c t + \phi_c(t)}_{\theta_i(t)})$$

where $\theta_i(t)$ is the instantaneous angle of the modulated carrier signal. The instantaneous frequency of the carrier is given by

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta_i(t)$$

In angle modulation, the angle of the carrier $\theta_i(t)$ is modified according to the message $m(t)$. The two main approaches to angle modulation are

1. Phase Modulation (PM): Where the instantaneous angle is modified linearly by the message signal. That is

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

   where k_p is known as the phase sensitivity.

2. Frequency modulation (FM) where the instantaneous frequency is modified linearly by the message signal. That is

$$f_i(t) = f_c + k_f m(t)$$

where k_f is known as the frequency sensitivity. To obtain the instantaneous angle, we integrate the above expression with respect to time and multiply the result by 2π. That is

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau$$

We see that a frequency modulated signal is similar to a phase modulated signal with the message signal replaced by its integral.

2.1 Properties of Angle Modulation

Angle modulation has a number of important properties

1. Since the amplitude of the carrier is constant, the average transmitted power is given by $\frac{A_c^2}{2}$.
2. Angle modulation is nonlinear
3. Zero crossings are irregular
4. Angle modulation is less sensitive to additive noise than amplitude modulation. This improved noise performance is achieved at the expense of increase bandwith requirements and complexity of implementation

3 Frequency Modulation

As we have seen, the frequency modulated signal is given by

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right)$$

Due to the nonlinearity of the modulation process, the spectrum of the FM signal is not related in a simple way to the spectrum of the message. We consider the spectral analysis in the simple case when $m(t) = A_m \cos(2\pi f_m t)$. In this case,

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

where $\Delta f = k_f A_m$ is known as the frequency deviation and is the maximum deviation of the instantaneous frequency from the carrier frequency.

We have

$$\begin{aligned} \theta_i(t) &= 2\pi \int_0^t f_i(\tau) d\tau \\ &= 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta \sin(2\pi f_m t) \end{aligned}$$

Where $\beta = \frac{\Delta f}{f_m}$ is known as the modulation index and is measured in radians.

The FM signal is then given by

$$\begin{aligned} s_{FM}(t) &= A_c \cos\Big(2\pi f_c t + \beta \sin(2\pi f_m t)\Big) \\ &= A_c \cos(2\pi f_c t) \cos\Big(\beta \sin(2\pi f_m t)\Big) - A_c \sin(2\pi f_c t) \sin\Big(\beta \sin(2\pi f_m t)\Big) \end{aligned}$$

Depending on the size of β we have two cases

1. Narrow-band FM: when β is small compared to one radian
2. Wide-band FM: when β is large compared to one radian

3.1 Narrow-band FM

When β is small compared to one radian we have $\cos\Big(\beta \sin(2\pi f_m t)\Big) \approx 1$ and $\sin\Big(\beta \sin(2\pi f_m t)\Big) \approx \beta \sin(2\pi f_m t)$ and

$$\begin{aligned} s_{FM}(t) &\approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \\ &= A_c \cos(2\pi f_c t) + \frac{\beta A_c}{2}\Big(\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\Big) \end{aligned}$$

This expression is similar to that obtained for conventional AM where

$$\begin{aligned} s_{AM}(t) &= A_c[1 + k_a A_m \cos(2\pi f_m t)] \cos(2\pi f_c t) \\ &= A_c \cos(2\pi f_c t) + \mu A_c \cos(2\pi f_c t) \cos(2\pi f_m t) \\ &= A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c\Big(\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\Big) \end{aligned}$$

where $\mu = k_a A_m$ is the modulating factor of the AM signal. The bandwidth of the narrow-band FM signal is therefore $2f_m$.

3.2 Wide-band FM

For an arbitrary value of β we have

$$\begin{aligned} s_{FM}(t) &= A_c \cos\Big(2\pi f_c t + \beta \sin(2\pi f_m t)\Big) \\ &= \mathrm{Re}\Big(A_c e^{j2\pi f_c t} e^{j\beta \sin(2\pi f_m t)}\Big) \end{aligned}$$

Since $\sin(2\pi f_m t)$ is periodic with period $1/f_m$, $e^{j\beta \sin(2\pi f_m t)}$ is also periodic with period $1/f_m$ and we can write it in terms of its Fourier series. We have

$$e^{j\beta \sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

Where

$$\begin{aligned} c_n &= f_m \int_0^{\frac{1}{f_m}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\beta(\sin u - nu)} du \end{aligned}$$

with the change of variables $u = 2\pi f_m t$. $\frac{1}{2\pi}\int_0^{2\pi} e^{j\beta(\sin u - nu)}du$ is the well known integral form of the *Bessel function of the First kind of order* n denoted by $J_n(\beta)$. Therefore we can write

$$e^{j\beta\sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta)e^{j2\pi n f_m t}$$

Also

$$\begin{aligned} s_{FM}(t) &= \text{Re}\Big(A_c e^{j2\pi f_c t} e^{j\beta\sin(2\pi f_m t)}\Big) \\ &= \text{Re}\Big(\sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi f_c t} e^{j2\pi n f_m t}\Big) \\ &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos\Big(2\pi(f_c + n f_m)t\Big) \end{aligned}$$

Taking the Fourier transform of the FM signal generated when the message signal $m(t) = A_m\cos(2\pi f_m t)$ we have

$$S_{FM}(f) = \sum_{n=-\infty}^{\infty} \frac{1}{2} A_c J_n(\beta)[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)]$$

From this expression we see that the bandwidth of this signal is infinite. However for large n the amplitude of the frequency components is small. For small β, we have

$$J_n(\beta) \approx \frac{\beta^n}{2^n n!}.$$

Only $J_0(\beta)$ and $J_1(\beta)$ have significant values and this corresponds to narrow-band FM

3.3 Transmission Bandwidth of FM Signals

We have seen that the FM signal generated with $m(t) = A_m\cos(2\pi f_m t)$ has an infinite bandwidth. In practice however only a finite number of frequency components have significant power. Therefore the effective bandwidth of the FM signal is finite. Frequency components more that Δf away from the carrier are effectively zero. For large β the effective bandwidth is $2\Delta f$ while for small β the effective bandwidth is $2f_m$. An empirical rule known as Carson's rule gives the effective transmission bandwidth for FM signals

$$B_T = 2\Delta f + 2f_m = 2\Delta f(1 + \frac{1}{\beta})$$

For an arbitrary message signal $m(t)$ with bandwidth W. We define the deviation ratio

$$D = \frac{\Delta f}{W}$$

D is equivalent to β encountered when $m(t) = A_m\cos(2\pi f_m t)$ a pure sinusoid. In this case Carson's rule becomes

$$B_T = 2\Delta f + 2W = 2\Delta f(1 + \frac{1}{D})$$

4 Generating FM signals

The generation of FM signals involves generation of frequencies in the output signal that are not present in the input signal. Therefore FM generators are not linear time invariant systems. FM signals are usually generated using time varying non-linear systems. A common way to directly generate FM signals is by using tuned oscillators with variable capacitance. If the capacitance is made to vary according to the message signal, the frequency of oscillation will also vary according to the message generating an FM signal. Consider a tuned LC circuit where

$$C(t) = C_0 + k_0 m(t)$$

When $m(t) = 0$ the oscillator's resonant frequency is equal to the carrier frequency. That is

$$f_c = \frac{1}{2\pi\sqrt{L_0 C_0}}$$

For non-zero $m(t)$ we have the instantaneous resonant frequency given by

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi\sqrt{L_0(C_0 + k_0 m(t))}} \\ &= \frac{1}{2\pi\sqrt{L_0 C_0}} \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}} \\ &= f_c \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}} \\ &\approx f_c(1 - \frac{k_0}{2C_0} m(t)) \end{aligned}$$

where we assume $\frac{k_0}{C_0} m(t) << 1$. This is the expression for a frequency modulated signal.