EEE 4107 Signals and Communication I.

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Today's Lecture

1. Linear systems
2. Convolution

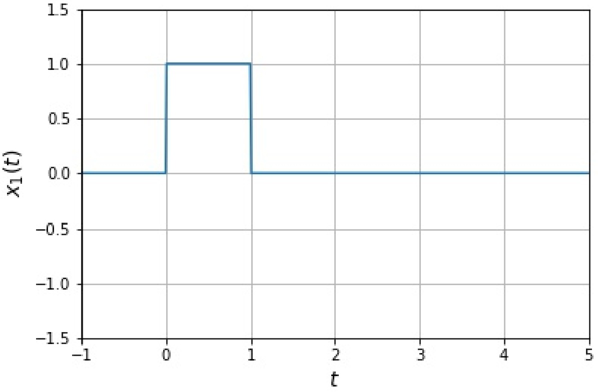
Linear Systems

Linear systems satisfy two properties namely

1. Superposition: if input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces output $y_2(t)$. Then the output of the system in response to input $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. Homogeneity: If input $x(t)$ produces output $y(t)$, then input $ax(t)$ where $a \in \mathbf{C}$ produces output $ay(t)$.

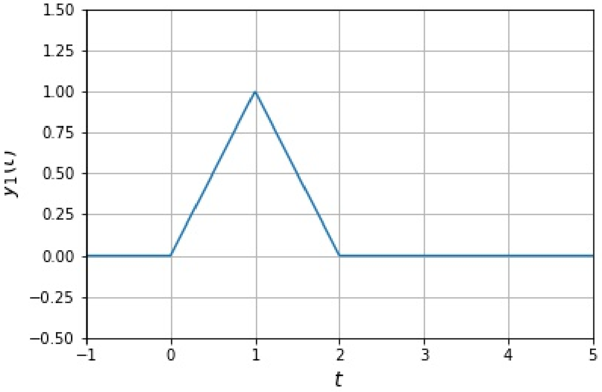
Homogeneity

- Consider the signal $x_1(t)$ shown below



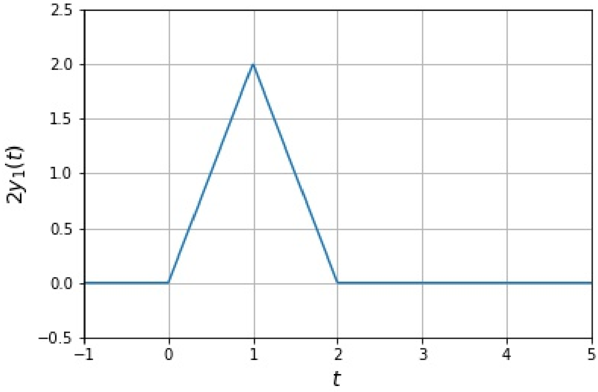
Homogeneity

- If the response of a linear system to $x_1(t)$ is $y_1(t)$ shown below



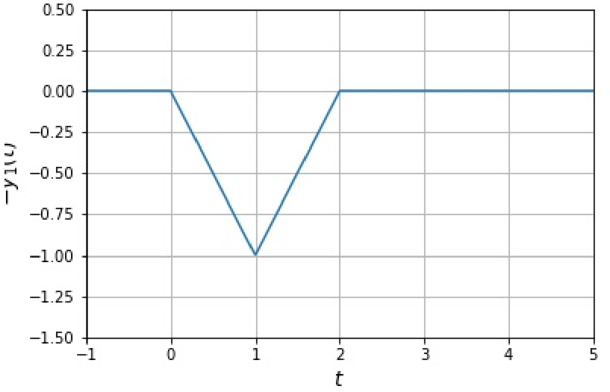
Homogeneity

- The response of a linear system to $2x_1(t)$ is $2y_1(t)$ shown below



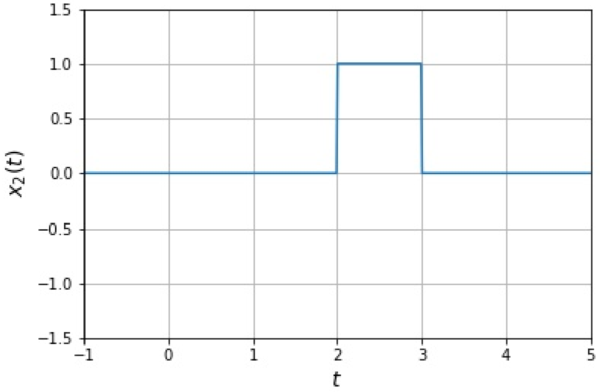
Homogeneity

- The response of a linear system to $-x_1(t)$ is $-y_1(t)$ shown below



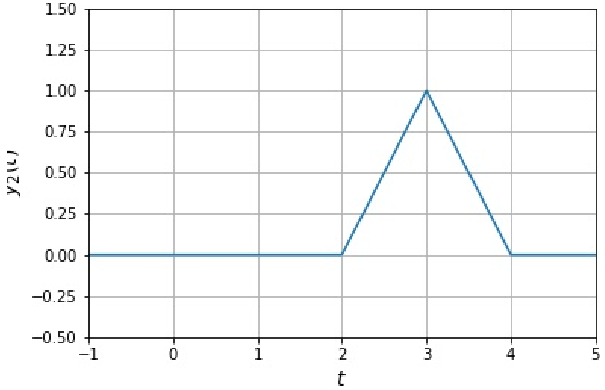
Superposition

- Now consider the signal $x_2(t)$ shown below



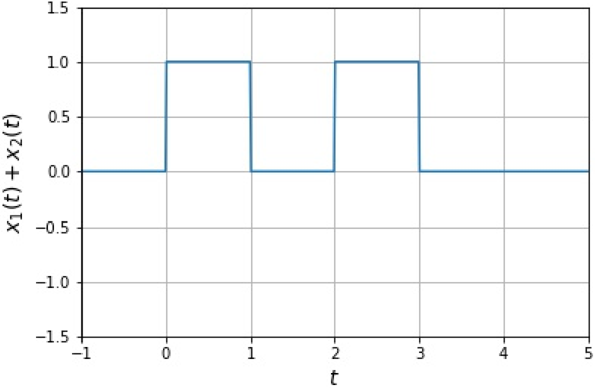
Superposition

- Assume the response of the same linear system to $x_2(t)$ is $y_2(t)$ shown below



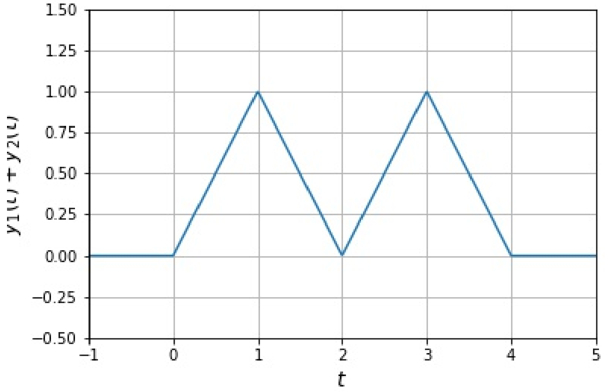
Superposition

- Now consider the signal $x_1(t) + x_2(t)$ shown below



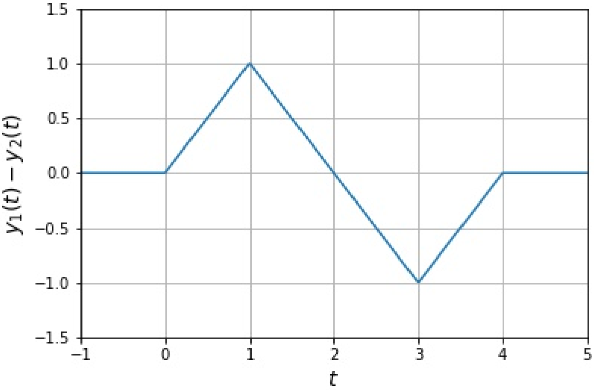
Superposition

- The response of the system to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$



Superposition

- Similarly the response of the system to $x_1(t) - x_2(t)$ is $y_1(t) - y_2(t)$

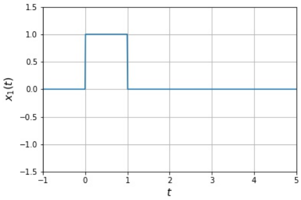


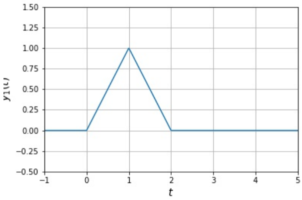
Time Invariance

- A system is said to be time invariant if a delay in the input produces the same delay in the output.
- Formally, if the response to $x(t)$ is $y(t)$, then the response to $x(t - D)$ is $y(t - D)$ where D is any real number.

Time Invariance

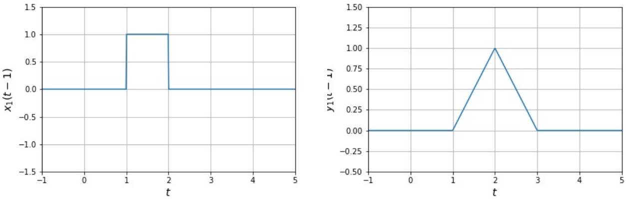
- Once again consider the signal $x_1(t)$ shown below. Assume the response of a time invariant system is given by $y_1(t)$





Time Invariance

- The response of a time invariant system to the signal $x_1(t-1)$ shown below



Convolution

- A system that is both linear and time invariant is known as a linear time invariant system.
- Several systems including communication systems can be modelled as linear time invariant (LTI) systems.
- An LTI system is completly characterised by its impulse response $h(t)$.
- The impulse response of an LTI system is the response of the system to the Dirac delta function $\delta(t)$.

Convolution

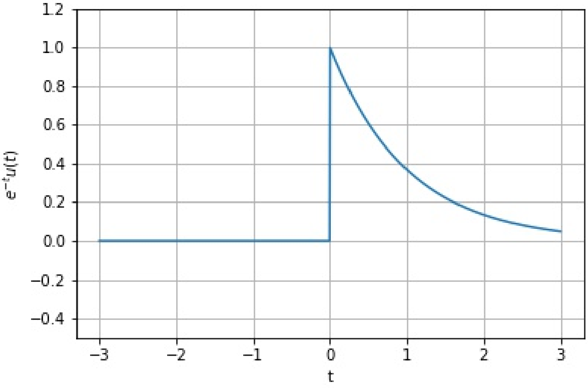
- In continuous time, the output $y(t)$ of an LTI system in response to input $x(t)$ is given by the convolution integral. That is
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (1)$$
- This is often denoted $y(t) = x(t) * h(t)$. The asterix denotes convolution not multiplication!
- Convolution is commutative. That is
$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad (2)$$

Convolution - Examples

- The impulse response of a linear time invariant system is given by $h(t) = e^{-t}u(t)$, where $u(t)$ is the unit step.
  - Sketch the response of the system to $x(t) = \delta(t)$
  - Determine and sketch the response of the systems to $u(t)$

Convolution - Examples

- Sketch the response of the system to $x(t) = \delta(t)$
  - Recall that the impulse respose $h(t)$ is the response of the system to the Dirac delta function $\delta(t)$
  - Therefore the response to $\delta(t)$ is $h(t) = e^{-t}u(t)$,



Convolution - Examples

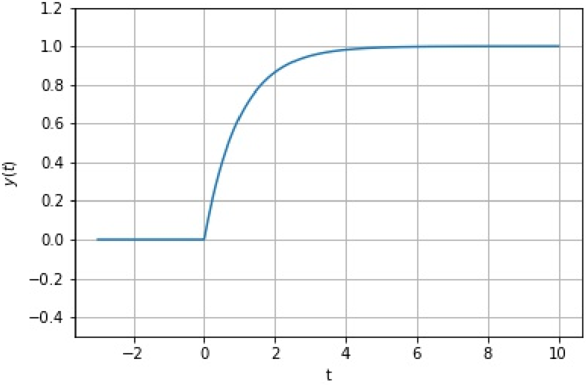
- Determine and sketch the response of the systems to $u(t)$
  - Recall that
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad (3)$$
  - $x(\tau) = u(\tau)$
  - $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$
  - $y(t) = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$
  - $y(t) = \int_{0}^{\infty} e^{-(t-\tau)}u(t-\tau)d\tau$ since $u(\tau)$ is zero for negative τ and 1 for positive τ
  - When $t < 0$, $u(t-\tau) = 0$ thus $y(t) = 0$
  - When $t > 0$
$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases} \qquad (4)$$
  - Therefore
$$y(t) = \int_{0}^{t} e^{-(t-\tau)}d\tau \qquad (5)$$

Convolution - Examples

- We get

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \geq 0 \end{cases} \tag{6}$$

- This can be written compactly as $y(t) = (1 - e^{-t})u(t)$



Convolution - Examples

- The impulse response of a linear time invariant system is given by $h(t) = e^{-t}u(t)$, where $u(t)$ is the unit step.
  - Determine and sketch the response of the systems to $u(t) - u(t-1)$

Convolution - Examples

- Recall that

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{7}$$

- $x(\tau) = u(\tau) - u(\tau - 1)$
- $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$
- $y(t) = \int_{-\infty}^{\infty} \big(u(\tau) - u(\tau-1)\big)e^{-(t-\tau)}u(t-\tau)d\tau$
- $y(t) = \int_{0}^{1} e^{-(t-\tau)}u(t-\tau)d\tau$
- When $t < 0$, $u(t-\tau) = 0$ thus $y(t) = 0$
- When $t > 0$

$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases} \tag{8}$$

Convolution - Examples

- Therefore

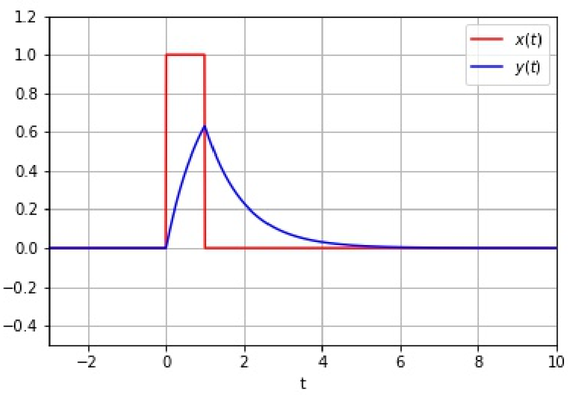
$$y(t) = \int_0^t e^{-(t-\tau)} d\tau \quad \text{for} \quad 0 < t < 1 \tag{9}$$

$$y(t) = \int_0^1 e^{-(t-\tau)} d\tau \quad \text{for} \quad t > 1 \tag{10}$$

- We get

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & 0 < t < 1 \\ (e-1)e^{-t} & t > 1 \end{cases} \tag{11}$$

Convolution - Examples



Convolution - Examples

- We can arrive at the above result by noting that the response to $u(t) - u(t-1)$ can be derived from the respose to $u(t)$
- We found that the response to $u(t)$ is

$$y_1(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \geq 0 \end{cases} \tag{12}$$

- Since the system is time invariant, the response to $u(t-1)$ is

$$y_2(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{-(t-1)} & t \geq 1 \end{cases} \tag{13}$$

- Since the system is linear, it satisfies superposition and homogeneity and the response to $u(t) - u(t-1)$ is $y_1(t) - y_2(t)$.

Convolution - Examples