EEE 4107 Signals and Communication I.

Dr. Ciira Maina
ciira.maina@dkut.ac.ke

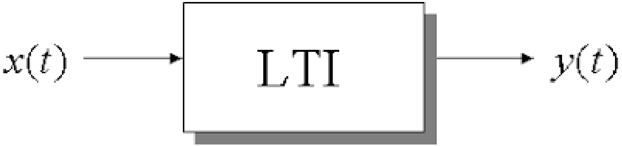
17th June, 2020

Today's Lecture

1. Convolution

Linear Time Invariant Systems

When analysing systems, we are interested in the response of a system to a given input.



LTI Systems

- For any continuous time signal $x(t)$, we can approximate it by

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta\tau) p_{\Delta\tau}(t - n\Delta\tau)\Delta\tau$$

where

$$p_{\Delta\tau}(t) = \begin{cases} \frac{1}{\Delta\tau} & |t| < \frac{\Delta\tau}{2} \\ 0 & \text{Otherwise} \end{cases}$$

See example in Notebook

LTI Systems

- If the response of the LTI system to $p_{\Delta\tau}(t)$ is $h_{\Delta\tau}(t)$, what is the response of the system to
  - $x(0)p_{\Delta\tau}(t)\Delta\tau$
  - $x(\Delta\tau)p_{\Delta\tau}(t-\Delta\tau)\Delta\tau$
  - $x(-\Delta\tau)p_{\Delta\tau}(t+\Delta\tau)\Delta\tau$
  - $x(-\Delta\tau)p_{\Delta\tau}(t+\Delta\tau)\Delta\tau + x(0)p_{\Delta\tau}(t)\Delta\tau + x(\Delta\tau)p_{\Delta\tau}(t-\Delta\tau)\Delta\tau$
  - $\sum_{n=-\infty}^{\infty} x(n\Delta\tau)p_{\Delta\tau}(t-n\Delta\tau)\Delta\tau$

LTI Systems

- If the response of the LTI system to $p_{\Delta\tau}(t)$ is $h_{\Delta\tau}(t)$, what is the response of the system to
  - $x(0)p_{\Delta\tau}(t)\Delta\tau \rightarrow x(0)h_{\Delta\tau}(t)\Delta\tau$
  - $x(\Delta\tau)p_{\Delta\tau}(t-\Delta\tau)\Delta\tau$
  - $x(-\Delta\tau)p_{\Delta\tau}(t+\Delta\tau)\Delta\tau$
  - $x(-\Delta\tau)p_{\Delta\tau}(t+\Delta\tau)\Delta\tau + x(0)p_{\Delta\tau}(t)\Delta\tau + x(\Delta\tau)p_{\Delta\tau}(t-\Delta\tau)\Delta\tau$
  - $\sum_{n=-\infty}^{\infty} x(n\Delta\tau)p_{\Delta\tau}(t-n\Delta\tau)\Delta\tau$

LTI Systems

- We have

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau) p_{\Delta\tau}(t - n\Delta\tau)\Delta\tau$$

Since as $\Delta\tau \to 0$, $p_{\Delta\tau}(t) \to \delta(t)$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

LTI Systems

- Since the response of the LTI system to $p_{\Delta\tau}(t)$ is $h_{\Delta\tau}(t)$

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau) h_{\Delta\tau}(t - n\Delta\tau)\Delta\tau$$

- As $\Delta\tau \to 0$, $h_{\Delta\tau}(t) \to h(t)$ and the sum tends to an integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- This is the convolution integral. Convolution is commutative

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Example

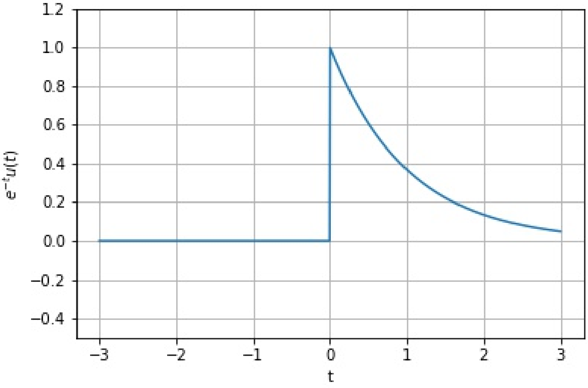
- Let $x(t) = u(t) - u(t-1)$ be the input to a system whose impulse response is $h(t) = u(t) - u(t-1)$. Compute the response.
- $$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
- See Notebook and video

Convolution - Examples

- The impulse response of a linear time invariant system is given by $h(t) = e^{-t}u(t)$, where $u(t)$ is the unit step.
  - Sketch the response of the system to $x(t) = \delta(t)$
  - Determine and sketch the response of the systems to $u(t)$

Convolution - Examples

- Sketch the response of the system to $x(t) = \delta(t)$
  - Recall that the impulse respose $h(t)$ is the response of the system to the Dirac delta function $\delta(t)$
  - Therefore the response to $\delta(t)$ is $h(t) = e^{-t}u(t)$,



Convolution - Examples

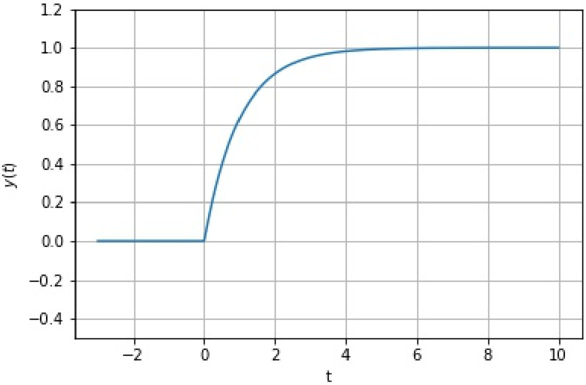
- Determine and sketch the response of the systems to $u(t)$
  - Recall that
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1}$$
  - $x(\tau) = u(\tau)$
  - $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$
  - $y(t) = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$
  - $y(t) = \int_{0}^{\infty} e^{-(t-\tau)}u(t-\tau)d\tau$ since $u(\tau)$ is zero for negative τ and 1 for positive τ
  - When $t < 0$, $u(t-\tau) = 0$ thus $y(t) = 0$
  - When $t > 0$
$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases} \tag{2}$$
  - Therefore
$$y(t) = \int_{0}^{t} e^{-(t-\tau)}d\tau \tag{3}$$

Convolution - Examples

- We get

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \geq 0 \end{cases} \tag{4}$$

- This can be written compactly as $y(t) = (1 - e^{-t})u(t)$
- See video



Convolution - Examples

- The impulse response of a linear time invariant system is given by $h(t) = e^{-t}u(t)$, where $u(t)$ is the unit step.
  - Determine and sketch the response of the systems to $u(t) - u(t-1)$

Convolution - Examples

- Recall that
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{5}$$
- $x(\tau) = u(\tau) - u(\tau - 1)$
- $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$
- $y(t) = \int_{-\infty}^{\infty} \big(u(\tau) - u(\tau-1)\big)e^{-(t-\tau)}u(t-\tau)d\tau$
- $y(t) = \int_{0}^{1} e^{-(t-\tau)}u(t-\tau)d\tau$
- When $t < 0$, $u(t-\tau) = 0$ thus $y(t) = 0$
- When $t > 0$
$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases} \tag{6}$$

Convolution - Examples

- Therefore

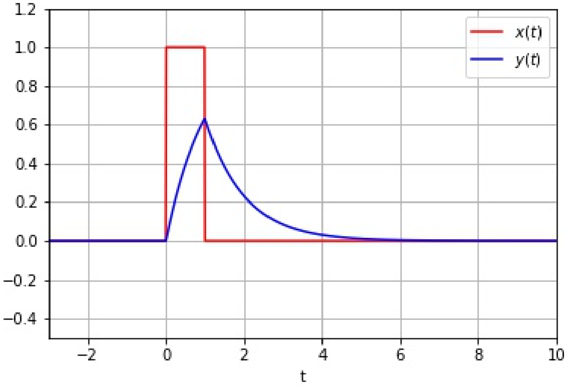
$$y(t) = \int_0^t e^{-(t-\tau)} d\tau \quad \text{for} \quad 0 < t < 1 \tag{7}$$

$$y(t) = \int_0^1 e^{-(t-\tau)} d\tau \quad \text{for} \quad t > 1 \tag{8}$$

- We get

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & 0 < t < 1 \\ (e-1)e^{-t} & t > 1 \end{cases} \tag{9}$$

Convolution - Examples



Convolution - Examples

- We can arrive at the above result by noting that the response to $u(t) - u(t-1)$ can be derived from the respose to $u(t)$
- We found that the response to $u(t)$ is

$$y_1(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \geq 0 \end{cases} \tag{10}$$

- Since the system is time invariant, the response to $u(t-1)$ is

$$y_2(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{-(t-1)} & t \geq 1 \end{cases} \tag{11}$$

- Since the system is linear, it satisfies superposition and homogeneity and the response to $u(t) - u(t-1)$ is $y_1(t) - y_2(t)$.

Convolution - Examples