EEE 4107 Signals and Communication I.

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Today's Lecture

1. Fourier Series

Introduction

- Communication systems can be modelled as LTI systems.
- In this case it is useful to model system inputs as linear combinations of basic signals.
- The superposition property of linear systems can then be used to obtain the output of the system.

Fourier Series

- A Periodic signal $x(t)$ with period T_0 can be expanded in terms of complex exponentials if it satisfies the Dirichlet conditions
- We have

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t}$$

where

$$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j\frac{2\pi n}{T_0}t}dt$$

and α is arbitrary.

Trigonometric representation of Fourier series

- For real valued signals, we can write the complex exponential Fourier series in terms of trigonometric functions. We have

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
a_n &= \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt \\
b_n &= \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt
\end{aligned}
$$

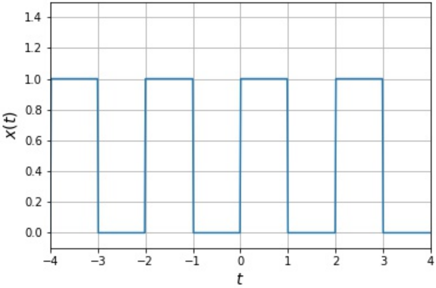
Trigonometric representation of Fourier series

- We can show that

$$x_n = \frac{a_n - jb_n}{2}$$

Example 1

- Consider the periodic signal below



- What is the period?

Example 1

- $T_0 = 2$
- We have

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t}$$

where

$$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j\frac{2\pi n}{T_0}t}dt$$

- Let $\alpha = 0$.
- $x_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\frac{2\pi n}{T_0}t}dt = \frac{1}{2}\int_0^2 x(t)e^{-j\pi nt}dt$
- $\frac{1}{2}\int_0^2 x(t)e^{-j\pi nt}dt = \frac{1}{2}\int_0^1 e^{-j\pi nt}dt$

Example 1

- Evaluating the integral we get

$$x_n = \frac{1}{2j\pi n}\left(1 - e^{-j\pi n}\right) \quad n \neq 0$$

- $x_0 = \frac{1}{2}$

Example 1 - Trigonometric Fourier Series

- We have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right) \right) \tag{1}$$

where

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt \\ b_n &= \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt \end{aligned}$$

Example 1 - Trigonometric Fourier Series

- Once again let $\alpha = 0$
- $a_n = \int_0^2 x(t) \cos\left(\pi n t\right) dt = \int_0^1 \cos\left(\pi n t\right) dt$
- Evaluating the integral we get $a_n = \frac{\sin(\pi n)}{\pi n} = 0$ for $n \neq 0$
- $a_0 = 1$
- Similarly we find $b_n = \frac{1-\cos(\pi n)}{\pi n}$ for $n = 1, 2, \ldots$
- $b_n = 0$ for even values of n
- Verify

$$x_n = \frac{a_n - jb_n}{2}$$

Example 1 - Trigonometric Fourier Series

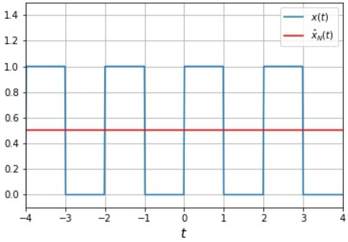
- Thus

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - \cos(\pi n)}{\pi n} \sin\left(\pi n t\right) \tag{2}$$

- We can form an approximation

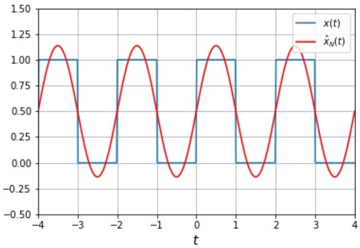
$$\hat{x}_N(t) = \frac{1}{2} + \sum_{n=1}^{N} \frac{1 - \cos(\pi n)}{\pi n} \sin\left(\pi n t\right) \tag{3}$$

- If we consider only the DC component ($N = 0$) we have



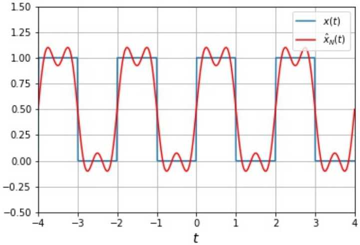
Example 1 - Trigonometric Fourier Series

- When $N = 1$ - The DC component and first sinusoidal component



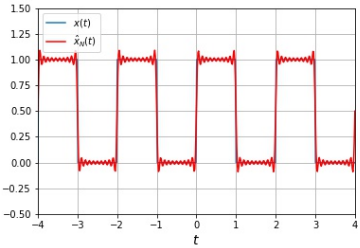
Example 1 - Trigonometric Fourier Series

- When $N = 4$



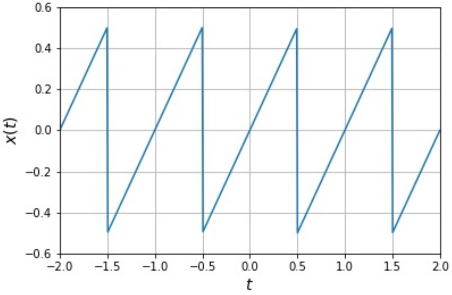
Example 1 - Trigonometric Fourier Series

- When $N = 20$



Example 2

- Consider the periodic signal below



- What is the period?

Example 2

- $T_0 = 1$
- We have

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t}$$

where

$$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j\frac{2\pi n}{T_0}t}dt$$

- Let $\alpha = -1/2$.
- $x_n = \int_{-1/2}^{1/2} x(t)e^{-j2\pi nt}dt$
- In the interval $-1/2 < t < 1/2$, $x(t) = t$
- Therefore $x_n = \int_{-1/2}^{1/2} te^{-j2\pi nt}dt$
- To evaluate the integral we use integration by parts

Example 2

- We have

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \tag{4}$$

- Use this to evaluate the integral for x_n

Example 2 - Trigonometric Fourier Series

- We have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right) \right) \quad (5)$$

where

$$a_n = \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt$$

$$b_n = \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt$$

Example 2 - Trigonometric Fourier Series

- Once again let $\alpha = -1/2$
- $a_n = 2 \int_{-1/2}^{1/2} t \cos\left(2\pi n t\right) dt$
- $b_n = 2 \int_{-1/2}^{1/2} t \sin\left(2\pi n t\right) dt$
- From integral tables we find

$$\int x \cos(ax) dx = \frac{\cos(ax)}{a^2} + \frac{x \sin(ax)}{a} \tag{6}$$

$$\int x \sin(ax) dx = -\frac{x \cos(ax)}{a} + \frac{\sin(ax)}{a^2} \tag{7}$$

see `http://integral-table.com/`

Example 2 - Trigonometric Fourier Series

- Evaluating the integral we get $a_n = 0$ for all n
- For odd signals, $a_n = 0$ for all n
- Similarly we find $b_n = \frac{-\cos(\pi n)}{\pi n}$ for $n = 1, 2, \ldots$
- Verify

$$x_n = \frac{a_n - jb_n}{2}$$

Example 2 - Trigonometric Fourier Series

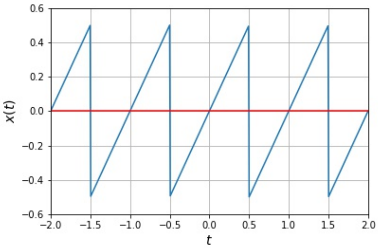
- Thus

$$x(t) = \sum_{n=1}^{\infty} \frac{-\cos(\pi n)}{\pi n} \sin\left(\pi n t\right) \tag{8}$$

- We can form an approximation

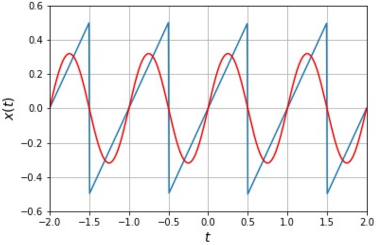
$$\hat{x}_N(t) = \sum_{n=1}^{N} \frac{-\cos(\pi n)}{\pi n} \sin\left(\pi n t\right) \tag{9}$$

- For $N = 0$ we have



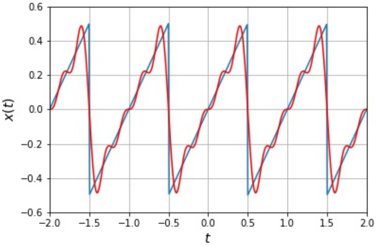
Example 2 - Trigonometric Fourier Series

- When $N = 1$



Example 2 - Trigonometric Fourier Series

- When $N = 4$



Example 2 - Trigonometric Fourier Series

- When $N = 20$